$\varphi(r, \tau)$, $v(r)$, $\psi(P, \tau)$	are the given functions;
D	is the problem domain;
S	is the surface bounding D;
B	is the domain congruent with D;
Ω	is the surface congruent with S;
F	is the surface exterior to B;
n	is the outward normal to a surface;
P	is the point on the surface S;
a, α, β, γ	are the parameters;
$G(r - r_0, \tau - t)$,	is the Green's function for unbounded space;
$\delta(r - r_0, \tau - t)$	is the Dirac delta function;
$q(r, \tau)$	is the function, nonvanishing only on the surface F.

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LINEAR DEFINING EQUATIONS IN HEAT-CONDUCTION THEORY WITH FINITE THERMAL-PERTURBATION VELOCITY

V. L. Kolpashchikov and A. I. Shnip

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A modification to the general theory of heat conduction with finite thermal-perturbation velocity, in which the linear defining equations are not thermodynamically forbidden is proposed.

In [1] Gurtin and Pipkin developed a general thermodynamic theory of heat conduction assuming propagation of the thermal perturbations at finite velocity. In the framework of this theory, they considered linear defining equations which lead to a linearized heat-conduction equation – in fact, an equation of hyperbolic type. However, the relation between the heat flux itself and the internal energy in this theory is not satisfied by the linear defining equations considered in [1], and therefore the resulting linearized heat-conduction may only be used with great inaccuracy, as a very rough guide.

The present paper outlines a modification of the Gurtin–Pipkin theory such that the linear defining equations (in fact, in terms of new independent variables) are not thermodynamically forbidden.

In the Gurtin–Pipkin theory the defining equations specify at some point **x** and time t the values of the free energy ψ, entropy η, and heat flux **q**, in terms of the temperature at time t, the total history of the temperature $\bar{\vartheta}^t$, and the total history of the temperature gradient $\bar{g}^t$

$$\begin{aligned} \psi &= \hat{\psi}(\vartheta, \bar{\vartheta}^t, \bar{g}^t), \\ \eta &= \hat{\eta}(\vartheta, \bar{\vartheta}^t, \bar{g}^t), \\ q &= \hat{q}(\vartheta, \bar{\vartheta}^t, \bar{g}^t). \end{aligned} \tag{1}$$

The total histories $\bar{\vartheta}^t$ and $\bar{g}^t$ are defined as follows:

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.

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$$\bar{\vartheta}^t(s) = \int_0^s \vartheta^t(\lambda)\,d\lambda, \quad \bar{g}^t(s) = \int_0^s g^t(\lambda)\,d\lambda, \tag{2}$$

where $\vartheta^t(\lambda) = \vartheta(t - \lambda)$ and $g^t(\lambda) = g(t - \lambda)$ are the histories of the temperature and the temperature gradient.

The internal energy e(t) is determined from the relation

$$e(t) = \psi(t) + \vartheta(t)\,\eta(t) = \hat{e}(\vartheta, \bar{\vartheta}^t \bar{g}^t). \tag{3}$$

Also in [1], the norm on the set of functions $\bar{\vartheta}^t(\cdot)$ and $\bar{g}^t(\cdot)$ which is the region of definition of the functionals in Eq. (1) was introduced, using some influence function $h(\cdot)$. In terms of this term, the Frechet continuity and continuous-differentiability conditions were formulated the required number of times for the functionals in Eq. (1). Under these assumptions Gurtin and Pipkin showed that Eq. (1) satisfies the second law of thermodynamics in the form of the Clausius–Duhem inequality

$$\dot{\psi} + \vartheta\eta + \frac{1}{\vartheta} g \cdot q \leqslant 0 \tag{4}$$

if and only if

a) the entropy and flux are completely determined from the free-energy functional using the relation

$$\hat{\eta}(\vartheta, \bar{\vartheta}^t, \bar{g}^t) = -D_\vartheta \hat{\psi}(\vartheta, \bar{\vartheta}^t, \bar{g}^t), \tag{5}$$

$$\hat{q}(\vartheta, \bar{\vartheta}^t, \bar{g}^t) = -\vartheta \delta_g \hat{\psi}(\vartheta, \bar{\vartheta}^t, \bar{g}^t | 1^+); \tag{6}$$

b) for all the permitted $\vartheta(\cdot)$ and $g(\cdot)$ the following dissipation relation holds

$$\delta_\vartheta \hat{\psi}(\vartheta, \bar{\vartheta}^t, \bar{g}^t | \vartheta^t - \vartheta(t)\,1^+) + \delta_g \hat{\psi}(\vartheta, \bar{\vartheta}^t, \bar{g}^t | g^t) \geqslant 0, \tag{7}$$

where D_ϑ is the differentiation operator with respect to ϑ, δ_ϑ and δ_g are the Frechet derivatives with respect to $\bar{\vartheta}^t$ and $\mathbf{g}^t$, respectively; 1^+ is a function whose value is 1 for all $s \in [0, \infty)$.

Operating on the left-hand and right-hand sides of Eqs. (3) and (5) with the operator δ_g and on Eq. (6) with the operator D_ϑ and combining Eqs. (5), (6), and (3) gives (since δ_g and D_ϑ are commutative)

$$D_\vartheta \hat{q}(\vartheta, \bar{\vartheta}^t, \bar{g}^t) = 2\,\frac{\hat{q}(\vartheta, \bar{\vartheta}^t, \bar{g}^t)}{\vartheta} + \delta_g \hat{e}(\vartheta, \bar{\vartheta}^t, \bar{g}^t). \tag{8}$$

Gurtin and Pipkin showed that for isotropic material the defining equations for the internal energy and heat flux should take the form

$$e(t) = e_0 + c\vartheta(t) + \int_0^\infty \alpha'(s)\,\bar{\vartheta}^t(s)\,ds, \tag{9}$$

$$g(t) = \int_0^\infty k'(s)\,\bar{g}^t(s)\,ds, \tag{10}$$

where e_0 is the measured internal energy; $\alpha(s)$ and $k(s)$ are relaxational functions; c = const is the instantaneous heat capacity. It is quickly evident that Eqs. (9) and (10) do not satisfy Eq. (8) for any nonzero heat flux and hence are thermodynamically forbidden.

In [2] the thermodynamic theory of heat conduction for solids with memory was somewhat modified by introducing a nonstandard set of thermodynamic variables. This gave a certain advantage in the analysis of the linear defining equations. It appears that an analogous modification in the case of the present theory would give an additional benefit: The linear defining equation would no longer be forbidden.

The independent variables chosen are the inverse temperature* $\theta = 1/\vartheta$, the total history of the inverse temperature $\bar{\theta}^t(\cdot)$, and the total history of the inverse temperature gradient $\bar{G}^t(\cdot)$. In addition, a new independent thermodynamic variable is introduced: the thermodynamic potential Φ, defined as follows:

$$\Phi = \frac{\psi}{\vartheta} = e\theta - \eta. \tag{11}$$

*Sometimes called the "coldness."

By analogy with Eq. (1), the defining equations for this case are

$$\Phi = \hat{\Phi}(\theta,\ \bar{\theta}^t,\ \bar{\mathbf{G}}^t),$$
$$e = \hat{e}(\theta,\ \bar{\theta}^t,\ \bar{\mathbf{G}}^t), \tag{12}$$
$$\mathbf{q} = \hat{q}(\theta,\ \bar{\theta}^t,\ \bar{\mathbf{G}}^t).$$

Using Eq. (11) and the definition $\theta = 1/\vartheta$, the Clausius–Duhem inequality in Eq. (4) may be transferred to give

$$e\dot{\theta} - \dot{\Phi} + \mathbf{q}\cdot\mathbf{G} \geqslant 0. \tag{13}$$

Then, retaining the assumption made by Gurtin and Pipkin with respect to the defining functionals in Eq. (12), it may be shown, by similar arguments, that Eq. (12) satisfies the Clausius–Duhem inequality in Eq. (13) if and only if:

a) The internal energy and the heat flux are completely determined from the thermodynamic-potential functional Φ by the relation

$$\hat{e}(\theta,\ \bar{\theta}^t,\ \mathbf{G}^t) = D_{\vartheta}\hat{\Phi}(\theta, \bar{\theta}^t,\ \bar{\mathbf{G}}^t), \tag{14}$$

$$\hat{q}(\theta,\ \bar{\theta}^t, \bar{\mathbf{G}}^t) = \delta_G \hat{\Phi}(\theta,\ \bar{\theta}^t,\ \bar{\mathbf{G}}^t); \tag{15}$$

b) for all the permitted $\theta(\cdot)$ and $\mathbf{G}(\cdot)$ the following dissipatove inequality holds

$$\delta_\theta \hat{\Phi}(\theta,\ \bar{\theta}^t,\ \bar{\mathbf{G}}^t | \theta^t - \theta(t)\, 1^+) + \delta_G \Phi(\theta,\ \bar{\theta}^t,\ \bar{\mathbf{G}}^t\, |\mathbf{G}^t) \leqslant 0. \tag{16}$$

Then the relation analogous to Eq. (8) for the present case is

$$D_\theta \hat{q}(\theta, \bar{\theta}^t,\ \bar{\mathbf{G}}^t) = \delta_G \hat{e}(\theta,\ \bar{\theta}^t,\ \bar{G}^t | 1^+). \tag{17}$$

For an isotropic material, as above, Eq. (12) reduces to the following:

$$e = \tilde{e}_0 + \tilde{c}\,\theta(t) + \int_0^\infty \tilde{\alpha}'(s)\,\bar{\theta}^t(s)\,ds, \tag{18}$$

$$\mathbf{q} = \int_0^\infty \tilde{k}'(s)\,\bar{\mathbf{G}}^t(s)\,ds, \tag{19}$$

which is always consistent with Eq. (17), in contrast to what was found with Eqs. (18), (19), and (10).

Substituting Eqs. (18) and (19) into the energy equations leads to a linearized heat-conduction equation of hyperbolic type, which is of the same form as the Gurtin–Pipkin equation but does not violate thermodynamics. It is only necessary to remember that this equation, unlike the Gurtin–Pipkin equation, describes the inverse-temperature field.

There is another difference between the modified theory and the Gurtin–Pipkin theory. In Gurtin–Pipkin theory, the calculation of the temperature-wave velocity gives (see Eqs. (6.8) and (6.9) in [1])

$$U = U_0 \sqrt{1 + m^2} + m), \tag{20}$$

where

$$U_0 = \sqrt{\frac{\mathbf{n}\cdot\mathbf{a}}{c}}, \tag{21}$$

$\mathbf{n}$ is the normal to the wavefront, and

$$\mathbf{a} = -\delta_g \hat{q}(\vartheta,\ \bar{\vartheta}^t,\ \bar{\mathbf{g}}^t | \mathbf{n}\, 1^+), \tag{22}$$

$$c = D_\vartheta \hat{e}(\vartheta,\ \bar{\vartheta}^t,\ \bar{\mathbf{g}}^t), \tag{23}$$

$$m = \frac{1}{U_0 c}\left\{\delta_g \hat{e}(\vartheta,\ \bar{\vartheta}^t,\ \bar{\mathbf{g}}^t | \mathbf{n}\, 1^+) + \frac{1}{\vartheta}\mathbf{g}\cdot\mathbf{n}\right\}. \tag{24}$$

For the modified theory, the temperature-wave velocity is again described by Eqs. (20) and (21), but in this case the definitions of a, c, and m are as follows:

$$a = -\delta_G \hat{g}(\theta, \bar{\theta}^t, \bar{G}^t|n\ 1^+), \tag{25}$$

$$c = D_\theta \hat{e}(\theta, \bar{\theta}^t, \bar{G}^t), \tag{26}$$

$$m = \frac{1}{U_0 c}\delta_g \hat{e}(\theta, \bar{\theta}^t, \bar{G}^t|n\ 1^+). \tag{27}$$

In [1], Eqs. (20)-(24) formed the basis for the conclusion that if

$$\delta_g e(\theta, \bar{\theta}^t, \bar{g}|n\ 1^+) = 0 \tag{28}$$

for all n, the temperature-wave velocity in the direction of **q** was larger than in the direction **−q**. Thus, this velocity is not simply a property of the material but is a function of the process. As follows from Eqs. (25)-(27), this effect is absent from the modified theory and, when only Eq. (28) is satisfied (if, for example, the material has a center of symmetry), $U = U_0$. The velocity U_0 may be regarded in the normal sense as a characteristic of the material since calculations of U_0 retaining only the main linear terms give a constant which depends solely on the temperature.

It should be emphasized that the difference of principle between the two theories are confirmed by experimental verification.

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HEAT TRANSFER IN SEMIINFINITE REGION WITH VARIABLE PHYSICAL PARAMETERS

Yu. I. Babenko

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A method is proposed for the determination of the nonsteady temperature field in a semiinfinite region with variable physical properties.

The heating of a semiinfinite region with variable physical parameters in the coordinate and the time, for zero initial conditions, may be described by the following equation

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \gamma(x, t)\right] T = 0, \quad 0 \leqslant x < \infty, \quad 0 < t < \infty, \tag{1}$$

$$T|_{x=0} = T_0(t); \quad T|_{x=\infty} = 0; \quad T|_{t=0} = 0.$$

It is required to find the temperature field T(x, t).

Earlier, for an analogous problem, only the temperature gradient at the boundary $(\partial T/\partial x)_{x=0}$ was found [1, 2].

The total solution of Eq. (1) will be sought in the form of a functional series

$$T = \sum_{n=0}^{\infty} c_n(x, t) D^{-n/2} e^{-xD^{1/2}} T_0(t). \tag{2}$$

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